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AN ELEMENTARY DERIVATION OF THE PRINCIPAL EFFECTS OF THE OBLATENESS OF THE EARTH UPON THE ORBITS OF SATELLITES

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ABSTRACT:

General formulas for the first-order effects of the oblateness of the Earth upon the orbits of satellites were reported in 1957 by R. E. Roberson and D. G. King-Hele. Their derivations employed methods which are unfamiliar to those not having studied the mathematics of astronomy. In this Technical Report / Research Paper No. 76, an elementary derivation is given for these useful and important results.

PRESENTATION:

This paper will be presented 26 May 1967 at the Canadian Congress of Applied Mechanics, Quebec.

An Elementary Derivation of the Principal Effects of the Oblateness of the Earth upon the Orbits of Satellites

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Independently of each other, in 1957 R. E. Roberson in the United States and D. King-Hele in Great Britain provided general formulas giving the first order effects of the oblateness of the Earth upon satellite orbits, previous analyses by others having contained inconvenient restrictions. Roberson and King-Hele obtained their results by the use of methods which are familiar to those who are trained in the mathematics of Astronomy but not familiar to those having only the usual training in Engineering Mechanics, who, accordingly, do not have easy access to these useful and important results.

In the present paper these results are derived by a quite elementary procedure requiring nothing more than elementary geometry, vector algebra, and mechanics. An appendix considers a few simplified models of the Earth which have the same first order effects as does an oblate Earth. This material should be of interest to engineers engaged in the space effort as a means of understanding the primary perturbations which result from oblateness.

A complete table of notation is given in Appendix B. This table is not arranged alphabetically, however, since so many of the definitions of the quantities depend on others in the list. Instead the arrangement is in "operational sequence", that is, in the order in which the symbols logically occur in the development. The notation agrees generallywith that used in astronautical literature; for example, V means geopotential, ν means true anomaly, and v denotes satellite (vector) velocity.

First we establish some coordinate systems. O is the mass center of the Earth and n is a unit vector parallel to the axis of rotation of the Earth pointing North. a and b are fixed unit vectors in the Earth's equatorial plane so as to form a right handed cartesian system a, b, n; actually, of course, because of the precession of the equinoxes and other slow motions, these vectors are not truly fixed, but we may regard them as being fixed for the purposes of this analysis. We could take a to be in the direction of the projection upon the Earth's equatorial plane of the line from O to the first point of Aires, but there seems to be no uniformity in this matter. We observe that conventionally the orbits of Earth satellites are referred to the Earth's equatorial plane in the same way that planetary orbits are referred to the plane of the ecliptic, but there does not seem to be perfect uniformity with regard to a reference position for longitude such as that provided by the first point of Aires.

Let Q be the instantaneous position of the satellite, with $\mathbf{r} = \vec{OQ}$ and $\mathbf{v} = \dot{\mathbf{r}}$, the dot denoting differentiation with respect to time. We define the scalar coordinates of satellite position by the equations

$$x = \mathbf{r} \cdot \mathbf{a}, \quad y = \mathbf{r} \cdot \mathbf{b}, \quad z = \mathbf{r} \cdot \mathbf{n} \qquad [1a,b,c]$$

so that

$$\mathbf{r} = x\mathbf{a} + y\mathbf{b} + z\mathbf{n} \qquad [2]$$

The geopotential for an oblate Earth (see Appendix A) may be approximated by the expression

$$V = -\mu r^{-1}[1 + Jr^{-2}(1 - 3z^2r^{-2})/3] \qquad [3]$$

where $\mu = GM_E$ is the product of the universal constant of gravitation by the mass of the Earth, J is the coefficient of the second zonal harmonic, and $r = \sqrt{\mathbf{r}\cdot\mathbf{r}}$. An approximate numerical value for μ is 9.66×10^4 (statute miles)^3(seconds)$^{-2}$. At the surface of the Earth, at the equator where r = 3964 statute miles, the product Jr^{-2} has the value 1.623×10^{-3} (dimensionless). Thus, evidently, for an Earth satellite, the second term in the square brackets in Eq. [3] is much smaller than unity. For a further discussion of Eq. [3], see Appendix A.

The force, per unit mass of satellite, exerted on the satellite by the gravitational attraction of the Earth is

$$\mathbf{f} = -\text{ grad } V = -\mathbf{a}\frac{\partial V}{\partial x} - \mathbf{b}\frac{\partial V}{\partial y} - \mathbf{n}\frac{\partial V}{\partial z} \qquad [4]$$

and to evaluate this expression, we note first that $r^2 = x^2 + y^2 + z^2$ so that

$$\frac{\partial r^n}{\partial x} = \frac{dr^n}{dr^2}\cdot\frac{\partial r^2}{\partial x} = \frac{d(r^2)^{n/2}}{dr^2}\cdot\frac{\partial r^2}{\partial x}$$
$$= (n/2)(r^2)^{n/2-1}(2x) = nxr^{n-2} \qquad [5]$$

with similar formulas in which y and z replace x. Thus

$$\mathbf{f} = \mu\,\text{grad}[r^{-1} + Jr^{-3}/3 - Jz^2r^{-5}]$$
$$= \mu\,[(-xr^{-3} - Jxr^{-5} + 5Jxz^2r^{-7})\mathbf{a} +$$
$$(-yr^{-3} - Jyr^{-5} + 5Jyz^2r^{-7})\mathbf{b} +$$
$$(-zr^{-3} - Jzr^{-5} + 5Jzz^2r^{-7} - 2Jzr^{-5})\mathbf{n}]$$

and by using Eq. [2] this may be more briefly written as

$$f = -\mu r^{-3}r - J\mu r^{-5}r(1 - 5z^2r^{-2}) - 2J\mu z r^{-5}n$$

$$= -\mu r^{-2}e_r - AJ\mu e_r - BJ\mu n \qquad [6]$$

where $e_r = r/r$ is a unit vector parallel to r and

$$A = r^{-4}(1 - 5z^2r^{-2}), \quad B = 2zr^{-5} \qquad [7a,b]$$

Note that if J were zero in Eq. [7], we would have the simple inverse square law of attraction toward the mass center of the Earth.

We now define h to be the angular momentum (with respect to O) of a unit mass of satellite, i. e.,

$$h = r\times v \qquad [8]$$

Since

$$v = \dot{r} = d(re_r)/dt = \dot{r}e_r + r\dot{e}_r$$

and since $e_r \cdot \dot{e}_r = 0$, we have

$$r\times h = r\times(r\times v) = r\cdot vr - r\cdot rv = r^2(e_r\cdot ve_r - v)$$

$$= r^2[(\dot{r} + 0)e_r - \dot{r}e_r - r\dot{e}_r] = -r^3\dot{e}_r \qquad [9]$$

We now use the fundamental principle of Mechanics that the rate of change of angular momentum (about O which we consider to be a fixed point since we are concerned with the motion of Q relative to O) is equal to the moment about O of the applied force; that is

$$\dot{h} = r\times f \qquad [10]$$

In the present case with only one moving particle, this formula is very easy to obtain from Eq. [8], from which

$$\dot{h} = \dot{r}\times v + r\times\dot{v} = r\times\dot{v} = r\times f$$

since $\dot{v}$ is the acceleration which is equal to f, the force per unit mass.

Next, consider the expression v×h. Taking the derivative with respect to time and using the notation $e_h = h/h$, where $h = \sqrt{h\cdot h}$, we have

$$\frac{d}{dt}(v\times h) = \dot{v}\times h + v\times\dot{h} = f\times h + v\times(r\times f)$$

$$= -\mu r^{-3}\mathrm{r}\times\mathrm{h} - J\mu(A\mathrm{e}_r + B\mathrm{n})\times\mathrm{h} + \mathrm{v}\times[\mathrm{r}\times(-BJ\mu\mathrm{n})]$$

$$= \mu\dot{\mathrm{e}}_r + \mu J h[A\mathrm{e}_h\times\mathrm{e}_r + B\mathrm{e}_h\times\mathrm{n} + Bh^{-1}\mathrm{v}\times(\mathrm{n}\times\mathrm{r})] \qquad [11]$$

Integrating this expression with respect to time gives

$$\mathrm{v}\times\mathrm{h} = \mu(\mathrm{e}_r + \mathrm{s}) \qquad [12]$$

where s is a vector whose time-derivative satisfies

$$\dot{\mathrm{s}} = Jh[A\mathrm{e}_h\times\mathrm{e}_r + B\mathrm{e}_h\times\mathrm{n} + Bh^{-1}\mathrm{v}\times(\mathrm{n}\times\mathrm{r})] \qquad [13]$$

Also, from Eq. [10] and Eq. [6],

$$\dot{\mathrm{h}} = BJ\mu\mathrm{n}\times\mathrm{r} \qquad [14]$$

Since we have seen, in Eq. [3], that the terms involving J are small compared to those not containing J, we regard J as a "small" quantity. Thus, Eq. [13] and Eq. [14] indicate that s and h are "nearly" constant vectors; that is, their magnitudes and orientations are but slowly changing.

Next, consider the scalar quantity

$$\mathrm{r}\cdot\mathrm{v}\times\mathrm{h} = \mathrm{r}\times\mathrm{v}\cdot\mathrm{h} = h^2 = \mu(r + \mathrm{r}\cdot\mathrm{s})$$

$$= \mu r(1 + \mathrm{e}_r\cdot\mathrm{s}) = \mu r(1 + s\cos v) \qquad [15]$$

where $s = \sqrt{\mathrm{s}\cdot\mathrm{s}}$ and $\cos v = \mathrm{e}_r\cdot\mathrm{s}/s$. v is the angle, shown in Fig. 1, which is known as the true anomaly. If we define the quantity

$$p = h^2\mu^{-1} \qquad [16]$$

which has the dimensions of a length, we may write

$$p = r(1 + s\cos v) \qquad [17]$$

In Eq. [17] only r and v vary "rapidly", $\dot{p} = 2h\mu^{-1}\dot{h}$ and $\dot{s}$ both involving J as a factor. Eq. [15] is that of an ellipse, called the instantaneous elliptical orbit (IEO), from which the actual orbit of the satellite begins to differ as time goes on. The point on this IEO (point P in Fig. 1) corresponding to $v = 0$ is called the instantaneous perigee point, toward which the unit vector $e_s = S/s$ points. Also, the eccentricity, s, of the IEO is called the instantaneous eccentricity. The angle i ($0 \leq i \leq \pi$), satisfying $\cos i = n \cdot e_h$, is the (instantaneous) inclination of the orbit plane. A unit vector in the direction $e_h \times n$, i. e., the vector $j = e_h \times n \csc i$, being perpendicular to both the plane of the IEO and the equatorial plane, thus points toward the descending node D (Fig. 1) of the former; $-j$ points toward the ascending node A. The angle $\omega = \angle AOP$ is called the argument of perigee. The quantity p is seen to be the semi-latus rectum of the IEO.

We will use the unit vector $i = j \times e_h$ such that i, j, e_h form a right handed triad, slowly changing in orientation since j and e_h are not constant. We let u and w be the angles BOQ and BOP, respectively (Fig. 1) so that $\cos u = i \cdot e_r$, $\cos w = i \cdot e_s$. Note that $u = v + w$ and that w is "nearly" constant since the IEO changes only slowly. This implies that $\dot{u} = \dot{v}$, approximately, a result which will be used repeatedly. Note also that $\omega = w + \pi/2$. We also introduce the unit vector $e_\theta = e_h \times e_s$ so that e_s, e_θ, e_h also form a slowly moving unit triad. Finally, we introduce the unit vector $I = j \times n$ so that I, j, n is a unit triad which rotates slowly about n. Thus, in particular, we have

$$i = I \cos i + n \sin i \qquad [18]$$

$$e_r = i \cos u + j \sin u \qquad [19]$$

A little manipulation easily establishes the following relations.

$$n \times r = (-I \sin u + j \cos i \cos u) \qquad [20]$$

$$z = r e_r \cdot n = r \sin i \cos u \qquad [21]$$

Now it is obvious that, to the approximation with which we are working, we have

$$\dot{e}_r = \dot{u} e_h \times e_r \qquad [22]$$

but it is also easily possible to obtain this result formally as follows

$$\dot{e}_r = \dot{i} \cos u + \dot{j} \sin u - (i \sin u - j \cos u)\dot{u}$$

$$\doteq (j \cos u - i \sin u)\dot{u}$$

since $\dot{i}$ and $\dot{j}$ nearly vanish. However,

$$e_h \times e_r = e_h \times j \sin u + e_h \times i \cos u = -i \sin u + j \cos u$$

and Eq. [22] is established. From this we have

$$h = r \times \dot{r} = r \times (\dot{r} e_r + r\dot{u} e_h \times e_r) = r^2 \dot{u} e_r \times (e_h \times e_r) = r^2 \dot{u} e_h$$

$$h = r^2 \dot{u} \qquad [22a]$$

$$r^{-3} = r^{-1} h^{-1} \dot{u} = p^{-1} h^{-1} \dot{u}(1 + s \cos v) \qquad [23]$$

Substituting Eqs. [7b], [20], [21], and [23] into Eq. [14], expanding $\cos v = \cos(u - w)$, and multiplying by dt, we obtain

$$dh = 2J\mu p^{-1} h^{-1} \sin i \,(1 + s \cos u \cos w + s \sin u \sin w)$$

$$\times(-I \sin u \cos u + j \cos i \cos^2 u) du \qquad [24]$$

It is elementary but tedious to obtain the indefinite integral of this expression and since the result of such an exercise does not seem to be of particular value, we will perform the integration for one complete passage only, as u increases by 2π, to obtain what we designate as Δh, the net change in h per passage. To do this most easily, note that for non-negative integers M, N

$$\int_0^{2\pi} \sin^M x \cos^N x \, dx = 0 \qquad [25]$$

unless *both* M and N are even. We easily obtain

$$\Delta h = jJ\mu\pi h^{-1} p^{-1} \sin 2i \qquad [26]$$

Since $j \cdot e_h = 0$, the variation, Δh, in the magnitude of h is of higher order in J. Thus we also have

$$\Delta e_h = jJ\mu\pi h^{-2} p^{-1} \sin 2i = jJ\pi p^{-2} \sin 2i \qquad [27]$$

using Eq. [16], and from the definition of j we have

$$\Delta j = \Delta(e_h \times n \csc i) = \Delta e_h \times n \csc i + e_h \times n \,\Delta \csc i$$

$$\Delta \mathrm{j} = 2J\pi p^{-2}\mathrm{j}\times\mathrm{n} \cos i + \mathrm{j}\Delta \csc i = 2J\pi p^{-2}\mathrm{I} \cos i \qquad [28]$$

since evidently we must omit the term in j because j·j = 1. We may also see in another way that there is no j-component since the variation in i is of higher order in J, $\Delta\mathrm{h}$ being perpendicular to the plane in which angle i is measured. We observe that the line of nodes rotates (in a sense opposite to that of the satellite itself if $i < \pi/2$) with mean angular velocity

$$\mathrm{B}_N^* = -2\mathrm{n}J\pi p^{-2}P^{-1}\cos i \qquad [29]$$

where P is the orbital period, which differs only by a term of order J from the value

$$P = 2\pi a^{3/2}\mu^{-1/2} \qquad [30]$$

(a is the semimajor axis of ellipse) which would obtain if J were zero. We use the asterisk in this formula and later to indicate a *mean* value in the sense that we have indicated. Thus we can finally obtain the very important formula

$$\mathrm{B}_N^* = -\mathrm{n}C \cos i \qquad [31]$$

where

$$C = (g/r_E)^{1/2}(J/r_E^2)(r_E/a)^{7/2}(1 - s^2)^{-2} \qquad [32]$$

in which r_E represents the equatorial radius of the earth and $g = \mu r_E^{-2}$ represents the acceleration of gravity at the surface. We have used the following relations

$$2a = r_{(v=0)} + r_{(v=\pi)} = p/(1+s) + p/(1-s) = 2p/(1-s^2)$$

$$2J\pi p^{-2}P^{-1} = Ja^{-7/2}(1-s^2)^{-2}\mu^{1/2} = Ja^{-7/2}(1-s^2)^{-2}g^{1/2}r_E^{-1} = C$$

Returning now to Eq. [13], we wish to express all vectors in terms of the (slowly moving) triad e_s, e_θ, e_h. We have

$$e_r = e_s \cos v + e_\theta \sin v; \quad n = e_h \cos i + \mathrm{i} \sin i$$

$$\mathrm{i} = e_s \cos w - e_\theta \sin w; \quad v = \dot{r} = \dot{r}e_r + r\dot{u}e_h \times e_r$$

It is now a matter of straightforward, if tedious, substitution into Eq. [13]. Some intermediate evaluations are shown below.

$$e_h \times e_r = -e_s \sin v + e_\theta \cos v$$

$$e_h \times n = e_h \times \mathrm{i} \sin i = \sin i\,(e_s \sin w + e_\theta \cos w)$$

$$n \times e_r = \begin{vmatrix} e_s & e_\theta & e_h \\ \sin i \cos w & -\sin i \sin w & \cos i \\ \cos v & \sin v & 0 \end{vmatrix}$$

$$= -e_s \cos i \sin v + e_\theta \cos i \cos v + e_h \sin i \sin u$$

$$v \times (n \times r) = r\dot{r}e_r \times (n \times e_r) + r^2\dot{u}(e_h \times e_r) \times (n \times e_r)$$

$$= r\dot{r}(e_s \sin i \sin u \sin v - e_\theta \sin i \sin u \cos v + e_h \cos i)$$
$$+ r^2\dot{u} \sin i \sin u\,(e_s \cos v + e_\theta \sin v)$$

From Eq. [22a] we have $r^2\dot{u} = h$ and from Eq. [17] and Eq. [23] we get $r\dot{r} = r^3p^{-1}s\dot{v} \sin v = hs \sin v\,(1 + s \cos v)^{-1}$. Thus we can write

$$v \times (n \times r) = (1 + s \cos v)^{-1}h\{\sin i \sin u\,[e_s(s + \cos v) + e_\theta \sin v]$$
$$+ e_h s \sin v \cos i\}$$

We also note that we may write $Jhr^{-4} = Jp^{-2}\dot{v}(1 + s \cos v)^2$ by using Eq. [23] and Eq. [17] and the fact that $\dot{u} = \dot{v}$ approximately. Thus, upon substituting into Eq. [13], we find

$$\dot{s} = Jp^{-2}(1 + s\cos v)\{(1 - 5\sin^2 i \cos^2 u)(1 + s\cos v)(-e_s \sin v +$$
$$e_\theta \cos v) + 2(1 + s\cos v) \sin^2 i \cos u\,(e_s \sin w + e_\theta \cos w) +$$
$$2 \sin^2 i \sin u \cos u\,[e_s(s + \cos v) + e_\theta \sin v] +$$
$$s e_h \sin 2i \sin v \cos u\}\,\dot{v}$$

Multiplying by dt and expanding functions of u in terms of v and w, w being regarded as constant, gives an expression for dS which may be integrated for general v. However, it is the net change per period which is of greatest interest, and upon use of Eq. [25], we find that

$$\Delta \mathbf{s} = Js\pi p^{-2}[(2-3\sin^2 i)\mathbf{e}_\theta - \mathbf{e}_h \sin 2i \sin w] \qquad [33]$$

Since $\mathbf{s}\cdot\Delta\mathbf{s} = 0$, the variation of s is of higher order in J, and we have

$$\Delta \mathbf{e}_s = J\pi p^{-2}[(2-3\sin^2 i)\mathbf{e}_\theta - \mathbf{e}_h \sin 2i \sin w] \qquad [34]$$

For the argument of perigee, we have

$$-\cos\omega = \mathbf{j}\cdot\mathbf{e}_s = \sin w$$

$$\begin{aligned} -\cos(\omega+\Delta\omega) &= -\cos\omega + \Delta\omega \sin\omega \\ &= (\mathbf{j}+\Delta\mathbf{j})\cdot(\mathbf{e}_s+\Delta\mathbf{e}_s) \\ &= \mathbf{j}\cdot\mathbf{e}_s + \mathbf{j}\cdot\Delta\mathbf{e}_s + \Delta\mathbf{j}\cdot\mathbf{e}_s \\ &= \sin w + J\pi p^{-2}[(2-3\sin^2 i)\mathbf{j}\cdot\mathbf{e}_\theta + 2\mathbf{I}\cdot\mathbf{e}_s \cos i] \\ &= \sin w + J\pi p^{-2} \cos w\,(4 - 5\sin^2 i) \end{aligned}$$

Since $\sin\omega = \cos w$, we clearly have

$$\Delta\omega = J\pi p^{-2}(4 - 5\sin^2 i) \qquad [35]$$

Dividing by P, the time of passage, we have

$$\dot{\omega}^* = (C/2)(4 - 5\sin^2 i) \qquad [36]$$

for the *mean* rate of change of argument of perigee. From Eq. [31] we see that

$$\dot{\Omega}^* = -C\cos i$$

is the *mean* rate of change of the longitude of the ascending node. Combining this result with that of Eq. [36], we find that the *mean* rate of change of the longitude of perigee is

$$\dot{\tilde{\omega}}^* = (C/2)(4 - 5\sin^2 i - 2\cos i) \qquad [38]$$

where $\tilde{\omega}$, the longitude of perigee, is the "angle" $\tilde{\omega} = \Omega + \omega$ measured in two different planes.

The result given in Eq. [36] is sometimes incorrectly referred to as the average rate of rotation of the line of apsides; strictly speaking it is simply the average rate of change of argument of perigee. The average rate of rotation of the line of apsides really is simply

$$P^{-1}\Delta \mathbf{e}_s = (C/2)[(2 - 3\sin^2 i)\mathbf{e}_\theta - \mathbf{e}_h \sin 2i \sin \omega]$$

the magnitude of which is

$$B_A^* = (C/2)[(2 - 3\sin^2 i)^2 + (\sin 2i \sin \omega)^2]^{1/2} \qquad [39]$$

The difference between this result and the value given in Eq. [36] is known to astronomers as the difference between the draconitic and the inertial motion of the apsides (1).

It is of some interest to have a formula for the mean angular velocity of the $\mathbf{e}_s$, $\mathbf{e}_\theta$, $\mathbf{e}_h$ triad. It is not difficult to show that this is given by the expression

$$B_{\text{Triad}}^* = (C/2)[(2 - 3\sin^2 i)\mathbf{e}_h - \mathbf{i} \sin 2i] \qquad [40]$$

The results of principal interest are given by Equations [26] through [40]. These equations or their equivalents seem to have been first obtained by R. E. Roberson (2) and by D. King-Hele (3). Both of them also discuss variations in the orbital period from the normal value

$$P = 2\pi a^{3/2}\mu^{-1/2}$$

but this analysis seems to be of a higher order of difficulty than that presented here. There are obvious practical difficulties in even defining an orbital period. It could, for example, refer to the successive passages through the perigee point (of the IEO determined at perigee), or it could refer to successive passages of the descending node. The analysis leading to Eq. [12] has been adapted from a recent popular textbook (4).

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Appendix A.

Remarks about Eq. [3].

A large literature has been developed concerning the potential of various mass distributions. Much of it has been concerned with the problem of determining the potential at an exterior point of a homogeneous ellipsoid. This problem is surprisingly difficult, but if the figure is of revolution, the results can be expressed in elementary (even though quite complicated) terms. Among readily available treatments, that of Thomson and Tait (5) is quite readable even though it is quite old. In an editorial footnote added in 1912, H. Lamb refers to a history of the subject by Todhunter. The latter seems to have been reprinted recently, but the writer has not been able to find a copy (6).

However, the Earth is certainly *not* a homogeneous ellipsoid, and, indeed, the finer details of the mass distribution of the Earth are presently being revealed by sophisticated analysis of observations made on actual satellite orbits. Our purpose in this Appendix is merely to indicate the genesis of Eq. [3] and to present a very simple model which yields Eq. [3].

If O is the mass center of the mass distribution, Q the exterior point at which potential V is being evaluated, and T is the general location of mass element dm, we let r denote the distnce OQ, ρ denote the distance OT, α denote the cosine of the angle QOT, and β denote the ratio ρ/r. It is well known then that the potential may be expressed in a series of integrals (over the mass distribution) of Legendre polynomials in α, viz.:

$$V = -(G/r)\int \Sigma P_n(\alpha)\ \beta^n\ dm$$

where the sum runs from zero to infinity and the symbol $P_n(\ )$ denotes the Legendre polynomial of degree n. If the body is homogeneous and nearly spherical, with principal moments of inertia I_x, I_y, and I_z, this may be developed as

$$V = (-\mu/r)\{1+(2r^2M)^{-1}[I_x+I_y+I_z-3(a^2I_x+b^2I_y+c^2I_z)]+\ldots\}$$

where a, b, and c are the direction cosines of line OQ with respect to the principal axes (through O). If we take the z axis as northward, and consider an oblate spherical Earth with semimajor axis A and semiminor axis B, this becomes

$$V = -\mu r^{-1}[1 + \frac{A^2-B^2}{10r^2}(1 - 3z^2r^{-2}) + \ldots]$$

and the coefficient J in Eq. [3] may be identified with the quantity $(A^2-B^2)/10$. The deleted terms, indicated by ... in the two preceding equations indicate terms of higher order in r^{-1}. However, since the Earth is not a homogeneous spheroid, the value of J cannot be calculated from a knowledge of A = equatorial radius and B = polar radius, but, instead, must be inferred from observed perturbations of satellite orbits. The forgoing has given an explanation of the source of Eq. [3] without, however, offering a model of the Earth from which Eq. [3] may be derived.

Another point of view is given in what follows. The oblateness of the Earth may be regarded as resulting in a deficiency of mass near the poles as compared to a homogeneous spherical Earth. Thus, we are led to consider a model composed of one *positive* point mass M located at O and two equal *negative* point masses $(-m)$ located on the polar axis at equal distances d from O; see Fig. 2. The potential V at Q is given by $-V/G = Mr^{-1} - mr_1^{-1} - mr_2^{-1}$. Now $r^2 = x^2 + z^2$, so that we have

$$(r_{1,2})^2 = x^2+(z\pm d)^2 = r^2\pm 2zd+d^2 = r^2[1+r^{-2}(d^2\pm 2zd)]$$

where the upper (negative) sign refers to r_1 and the lower (positive) sign refers to r_2. Combining these relations, we find

$$Vr/Gm = -(M/m) + 2 - (d/r)^2(1 - 3z^2/r^2) + \ldots$$

plus terms involving higher powers of $(d/r)^2$. Since we propose to keep (d/r) quite small, we will be justified in neglecting them. Thus, writing $n = m/M$, a positive number, we have

$$V = - GMr^{-1}(1 - 2n) - nMGd^2r^{-3}(1 - 3z^2r^{-2}) + \ldots$$

Now $M(1-2n) = M-2m = M_E$, the mass of the Earth, so that $M = M_E/(1-2n)$. Thus, recalling that $\mu = GM_E$, we have

$$V = - \mu r^{-1}\{1 + [nd^2/(1-2n)]r^{-2}(1 - 3z^2r^{-2}) + \ldots\}$$

so that we identify $J/3$ with the quantity $nd^2/(1-2n)$. Thus

$$5.41\times10^{-4} = Jr_E^{-2}/3 = [n(1-2n)](d/r_E)^2$$

and

$$(d/r_E)^2 = 5.41\times10^{-4}(n^{-1}-2)$$

There is, of course, no unique decomposition. If we take $n = 1/3$, then $m = M_E$ and $M = 3M_E$, with $d = 0.023r_E = 92.3$ statute miles, and so on. The following tabulation gives a few such choices.

M/M_E	m/M_E	$(d/r_E)^2$	d (miles)
101	50	$1.08\cdot10^{-5}$	13.0
21	10	$5.41\cdot10^{-5}$	29.2
5	2	$2.71\cdot10^{-4}$	63.6
3	1	$5.41\cdot10^{-4}$	92.3
2	.5	$1.08\cdot10^{-3}$	130.4

Since these values of $(d/r_E)^2$ are so small, then, *a forteriori*, values of (d/r) will be even smaller, where r is the distance from the center of the earth to a satellite; thus, truncation of the binomial series expansions of $(r_{1,2})^{-1/2}$ is well justified and any of these models does a good job of representing the Earth's actual potential field.

Appendix B.

List of Notations.

Except for the division into general catagories (general, points, vectors, scalars), the listing is in the sequence that the symbols are introduced in the text rather than in alphabetical order.

A. General

Δ denotes net change (per orbit passage) of quantity following.

$\cdot$ $\frac{d}{dt}$ denotes differentiation with respect to time, t.

$*$ emphasizes *mean* value over one orbit passage.

B. Points

O = mass center of the Earth.

Q = position of satellite.

P = perigee point of IEO.

D = descending node of IEO.

A = ascending node of IEO.

B = point on orbit on vector $\mathbf{i}$ extended.

T = location of mass element dm.

C. Vectors

$\mathbf{n}$ = unit North vector.

$\mathbf{a}$,$\mathbf{b}$ = fixed unit vectors in Earth's equatorial plane.

$\mathbf{r}$ = vector OQ = position vector of satellite.

$\mathbf{v} = \dot{\mathbf{r}}$ = velocity of satellite

$\mathbf{f}$ = attractive force per unit mass of satellite.

$\mathbf{e}_r = \mathbf{r}/r$ = unit vector in direction of satellite position.

$\mathbf{h} = \mathbf{r}\times\mathbf{v}$ = angular momentum about O per unit mass of satellite.

$\mathbf{e}_h = \mathbf{h}/h$ = unit vector in direction of $\mathbf{h}$.

$\mathbf{s}$ = vector satisfying Eq. [13].

$\mathbf{e}_s = \mathbf{s}/s$ = unit vector from O toward P.

$\mathbf{j} = \mathbf{e}_h\times\mathbf{n} \csc i$ = unit vector from O toward D.

$\mathbf{i} = \mathbf{j}\times\mathbf{e}_s$ = unit vector of triad $\mathbf{i}$, $\mathbf{j}$, $\mathbf{e}_h$.

$\mathbf{e}_\theta = \mathbf{e}_h\times\mathbf{e}_s$ = unit vector of triad $\mathbf{e}_s$, $\mathbf{e}_\theta$, $\mathbf{e}_h$.

$\mathbf{I} = \mathbf{j}\times\mathbf{n}$ = unit vector of triad $\mathbf{I}$, $\mathbf{j}$, $\mathbf{n}$.

$\mathbf{B}_N^*$ = *mean* angular velocity of line of nodes.

$\mathbf{B}_{\mathrm{Triad}}^*$ = *mean* angular velocity of $\mathbf{e}_s$, $\mathbf{e}_\theta$, $\mathbf{e}_h$ triad.

D. *Scalars; dimensionality is indicated following semicolon.*

$x = \mathbf{r}\cdot\mathbf{a}$; L.

$y = \mathbf{r}\cdot\mathbf{b}$; L.

$z = \mathbf{r}\cdot\mathbf{n}$; L.

V = geopotential at point Q; L^2T^{-2}.

$\mu = GM_E \doteq 9.66\times10^4$(statute miles)^3(seconds)$^{-2}$; L^3T^{-2}.

G = coefficient of universal gravitation; $L^4F^{-1}T^{-4}$.

J = coefficient of second zonal harmonic of potential

$\doteq 1.623\times10^{-3}r_E^2$; L^2.

$r = \sqrt{\mathbf{r}\cdot\mathbf{r}}$ = distance OQ; L.

$A = r^{-4}(1 - 5z^2r^{-2})$; L^{-4}.

$B = 2zr^{-5}$; L^{-4}.

Also, in Appendix A, letters A *and* B *denote semi-axes of the Earth.*

t = time; T.

$h = \sqrt{\mathbf{h}\cdot\mathbf{h}}$ = magnitude of angular momentum per unit mass; L^2T^{-1}.

v = true anomaly, angle POQ = $\cos^{-1}(\mathbf{e}_s\cdot\mathbf{e}_r)$; dimensionless.

$p = h^2\mu^{-1}$ = semi-latus rectum of IEO; L.

$s = \sqrt{\mathbf{s}\cdot\mathbf{s}}$ = eccentricity of IEO; dimensionless.

i = inclination of plane of IEO with respect to equatorial plane

= $\cos^{-1}(\mathbf{n}\cdot\mathbf{e}_h)$; dimensionless.

ω = argument of perigee = angle AOP; dimensionless.

u = angle BOQ = $\cos^{-1}(\mathbf{i}\cdot\mathbf{e}_r)$; dimensionless.

w = angle BOP = $\cos^{-1}(\mathbf{i}\cdot\mathbf{e}_s)$; dimensionless.

a = semimajor axis of IEO; L.

$P = 2\pi a^{3/2}\mu^{-1/2}$ = period of orbital passage; T.

$C = (g/r_E)^{1/2}(J/r_E^2)(r_E/a)^{7/2}(1 - s^2)^{-2}$

= coefficient in several important results; T^{-1}.

r_E = radius of the Earth; L.

$g = \mu r_E^{-2}$ = acceleration of gravity at surface of Earth; LT^{-2}.

$\dot{\omega}^*$ = *mean* rate of change of argument of perigee; T^{-1}.

$\dot{\Omega}^*$ = *mean* rate of change of longitude of ascending node; T^{-1}.

$\dot{\tilde{\omega}}^*$ = *mean* rate of change of longitude of perigee; T^{-1}.

B_A^* = *mean* rate of rotation of line of apsides; T^{-1}.

ρ = distance OT to mass element dm; L.

α = cosine of angle QOT; dimensionless.

$\beta = \rho/r$; dimensionless.

$P_n(\ )$ = Legendre polynomial of degree n; dimensionless.

I_x, I_y, I_z = principal moments of inertia at O; FLT^2.

a, b, c = direction cosines of OQ with respect to principal axes; dimensionless.

A,B = semiaxes of oblate Earth; L.

M = *positive* mass at O; FT^2L^{-1}.

$-m$ = *negative* mass on polar axis; FT^2L^{-1}.

d = distance from M to $-m$; L.

$n = m/M$; dimensionless.

$M_E = M - 2m$ = mass of Earth; FT^2L^{-1}.

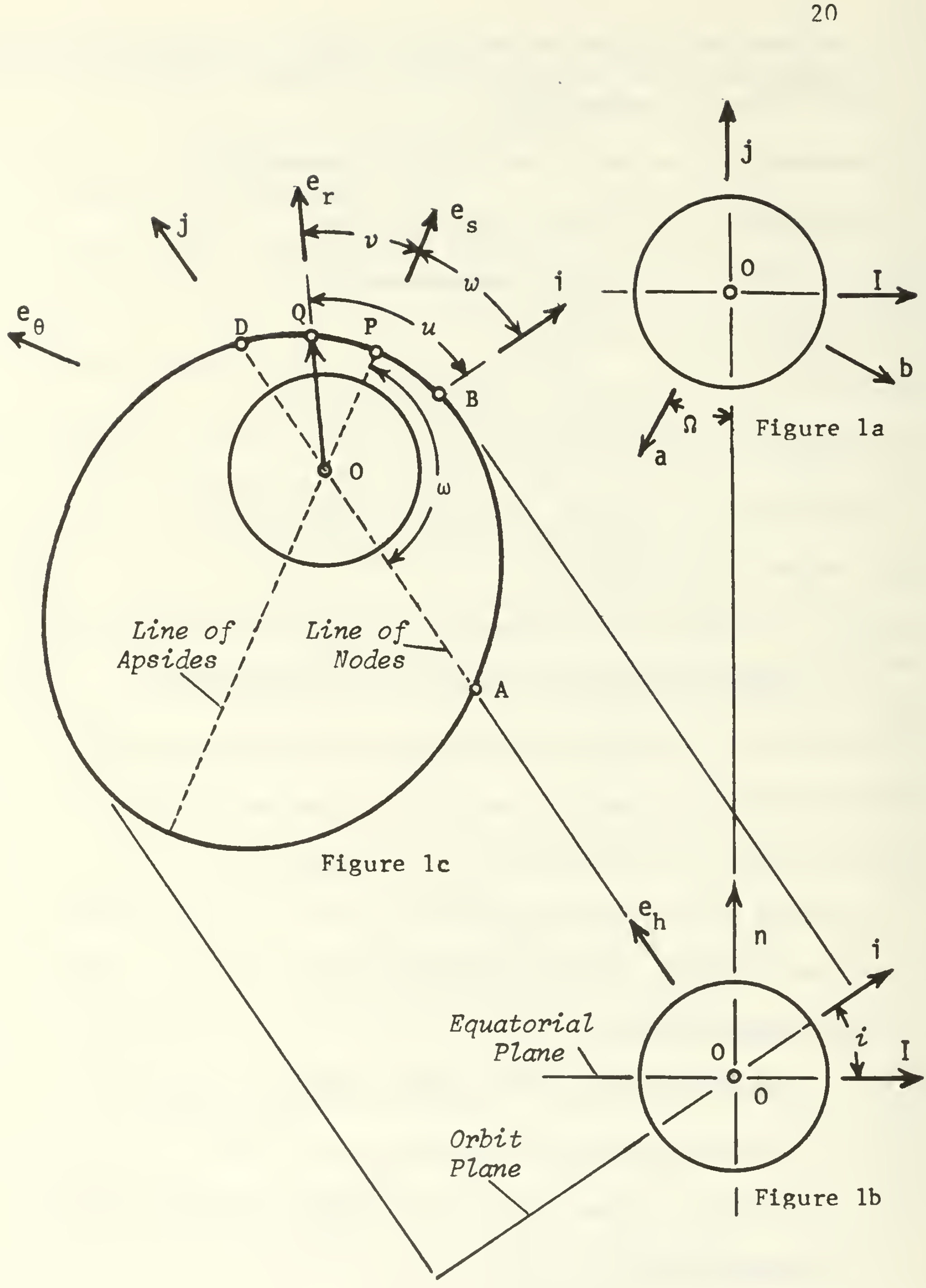
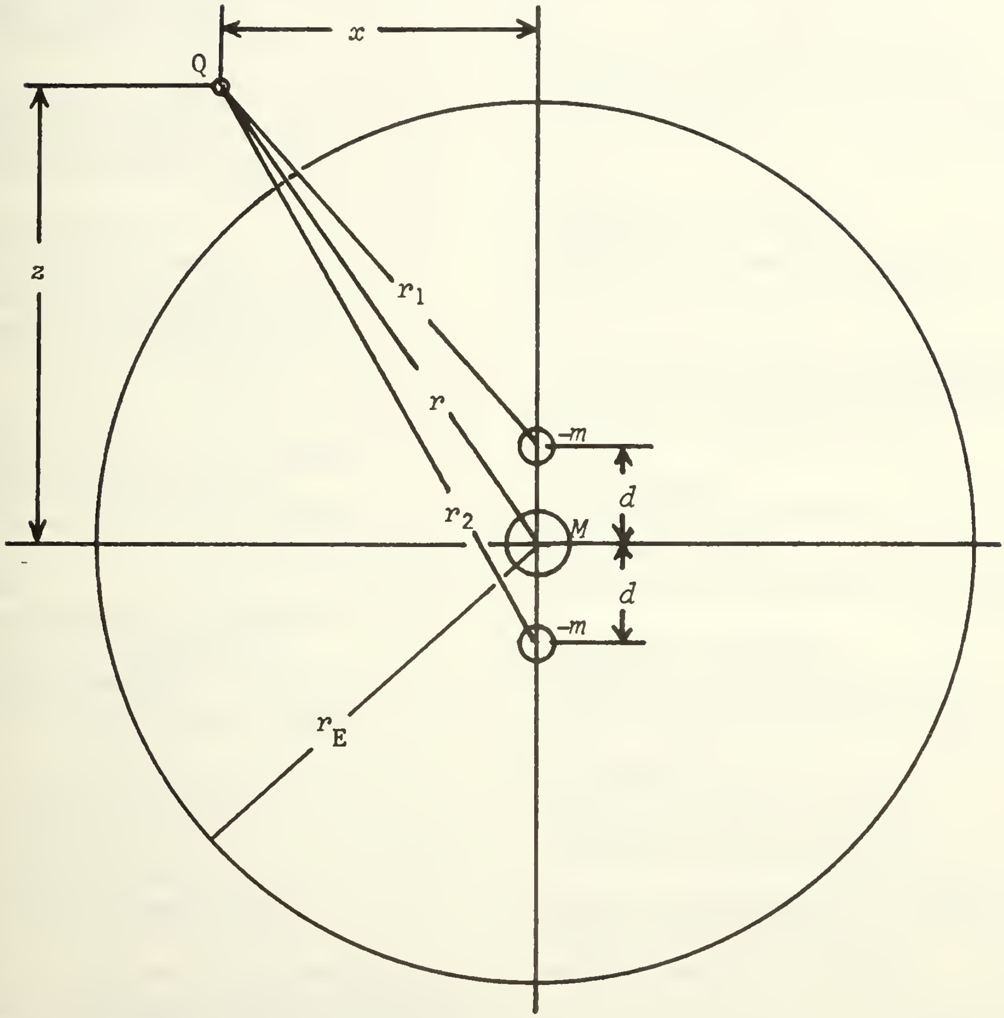


Figure 1 Details of orbit of Earth satellite. Figure 1a is looking south along axis of Earth; Figure 1b is looking along line of nodes from ascending node A toward descending node D; Figure 1c is true projection of orbit. P is the perigee point, O is the center of the Earth, and Q is the location of the satellite. Angle AOB is a right angle. Ω is the longitude of the ascending node, ω is the argument of perigee, ν is the true anomaly, and angles u and w are as indicated.

Figure 2 Simplified model of the Earth consisting of *positive* mass M at center and two *negative* masses $-m$ on polar axis.



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13. ABSTRACT

General formulas for the first-order effects of the oblateness of the Earth upon the orbits of satellites were reported in 1957 by R. E. Roberson and by D. G. King-Hele. Their derivations employed methods which are unfamiliar to those not having studied the mathematics of astronomy. In this Technical Report / Research Paper No. 76, an elementary derivation is given for these useful and important results.

14 KEY WORDS	LINK A		LINK B		LINK C	
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Earth						
Geopotential						
Node						
Oblate						
Oblateness						
Orbit						
Perigee						
Perturbation						
Potential						
Satellite						
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